ON THE FORMS OF UNICURSAL QUINTIC CURVES

DISSERTATION

PRESENTED TO THE UNIVERSITY FACULTY OF CORNELL UNIVERSITY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

PETER FIELD

June, 1902

BALTIMORE
The Lord Baltimore Press
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INTRODUCTION.

The purpose of this paper is to study the forms of plane unicursal quintic curves. The basis of classification is the same as that used by Meyer,* but the method of obtaining the curves is entirely different. Meyer classified the curves according to the sequence of the double points and his scheme was similar to that employed by Tait † in his first paper on knots. The essential difference is that nugatory knots may appear. Suppose a curve has the six double points A, B, C, D, E, F, and suppose a variable point to describe the curve. Whenever the generating point passes through a node, indicate this fact by writing the symbol of that node. When a loop is formed near a node A the symbol would be .... AA .... for that part. Two curves which have different symbols will be regarded as distinct. A more specific classification would be one which considers the reality and positions of the inflexions, but this would lead to an almost infinite variety of forms.

In this way Meyer first wrote the 29 possible schemes for nondegenerate curves, and gave a figure for each one. He gave no equations and did not extend the method to other types of quintic curves. In his second paper he corrected some errors made in the first and showed that all the types can be obtained by quadric inversion.

Dowling ‡ devotes an article in his dissertation to the unicursal quintic curves. He derives an equation of a different form from the one that will be used in this paper. He does not give any figures. These and the references given on p. 219 of Loria's Spezielle Algebraische Kurven are the only papers known to me which deal with plane unicursal quintic curves.

In a few cases, more than one figure will be given having the same sequence

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* (a) Anwendungen der Topologie auf die Gestalten der algebraischen Curven vierter und fünfter Ordnung. Muenchen (Diss.), 1878.

(b) Proceedings of the Edinburgh Royal Society, 1886. † Edinburgh Transactions, 1876–77.

‡ On the forms of plane quintic curves, Mathematical Review, Vol. 1.

of points but which nevertheless are so different in appearance as to deserve separate notice.

The following table gives Plücker's and Klein's numbers:

A is the number of real acnodes.

T the number of real double tangents with imaginary points of contact.

I the number of real inflexions.

K the number of real cusps.

n	δ	$\varkappa$	τ	i
8	6	0	12	9

A	T	I
0	0	3
	1	1
1	0	5
	1	3
	2	1
2	0	7
	1	5
	2	3
	3	1
3	0	9
	1	7
	2	5
	3	3
	4	1
4	1	9
	2	7
	3	5
	4	3
	5	1
5	2	9
	3	7
	4	5
	5	3
	6	1
6	3	9
	4	7
	5	5
	6	3
	7	1

n	δ	$\varkappa$	τ	i
7	5	1	8	7

A	T	I
0	0	3
	1	1
1	0	5
	1	3
	2	1
2	0	7
	1	5
	2	3
	3	1
3	1	7
	2	5
	3	3
	4	1
4	2	7
	3	5
	4	3
	5	1
5	3	7
	4	5
	5	3
	6	1

n	δ	$\varkappa$	τ	i
6	4	2	5	5

K	A	T	I
0	0	0	1
0	1	0	3
		1	1
0	2	0	5
		1	3
		2	1
0	3	1	5
		2	3
		3	1
0	4	2	5
		3	3
		4	1
2	0	0	3
		1	1
2	1	0	5
		1	3
		2	1
2	2	1	5
		2	3
		3	1
2	3	2	5
		3	3
		4	1
2	4	3	5
		4	3
		5	1

n	δ	$\varkappa$	τ	i
5	3	3	3	3

K	A	T	I
1	0	0	1
1	1	0	3
		1	1
1	2	1	3
		2	1
1	3	2	3
		3	1
3	0	0	3
		1	1
3	1	1	3
		2	1
3	2	2	3
		3	1
3	3	3	3

n	δ	$\varkappa$	τ	i
4	2	4	2	1

K	A	T	I
0	1	0	1
0	2	1	1
2	0	0	1
2	1	1	1
2	2	2	1
4	0	1	1
3	1	2	1

1.—*Quintic curves with six distinct nodes.*

The equation of the fifth degree contains twenty constants so that if the six nodal points are fixed but two additional conditions may be imposed in order to fix the curve. The general equation of such a curve will now be derived and the form of the curves determined by its aid.

Let 1, 2, 3, 4, 5, 6 be the six double points; let U_0 and U_1 be two nodal cubics through these points, U_0 having a node at 1 and U_1 at 2; further let ϕ_0 and ϕ_1 be two conics, the former through 2, 3, 4, 5, 6 and the latter through 1, 3, 4, 5, 6. Then the equation

$$\phi_0 U_0 - \lambda \phi_1 U_1 = 0\,,$$

is the equation of any quintic curve having the given points as double points. For the curve might be defined by giving another point and the tangent at the given point. But each of the curves U_0 and U_1 contains an arbitrary constant, hence the given point may be taken as their ninth intersection and the value of λ can be taken so as to give the desired slope.

The above equations might also be considered as representing the intersections of corresponding rays of the pencils $\lambda U_0 - U_1 t = 0$ and $\phi_0 - \phi_1 t = 0$, in which t is the parameter. The points 3, 4 and 5, 6 may be conjugate imaginary and the cubics and conics are nevertheless real, but in case all six of the double points are imaginary the above form of equation is no longer applicable. In that case suppose 1 and 2, 3 and 4, 5 and 6 are conjugate imaginary points. Let α_1 be a line through 1 and 2, α_2 through 3 and 4, α_3 through 5 and 6, also let

$$\phi_1 + i\lambda_1\psi_1,\quad \phi_1 - i\lambda_1\psi_1,\quad \phi_2 + i\lambda_2\psi_2,\quad \phi_2 - i\lambda_2\psi_2,\quad \phi_3 + i\lambda_3\psi_3,\quad \phi_3 - i\lambda_3\psi_3,$$

be conics through the points

1, 3, 4, 5, 6   2, 3, 4, 5, 6   1, 2, 4, 5, 6   1, 2, 3, 5, 6   1, 2, 3, 4, 6   1, 2, 3, 4, 5

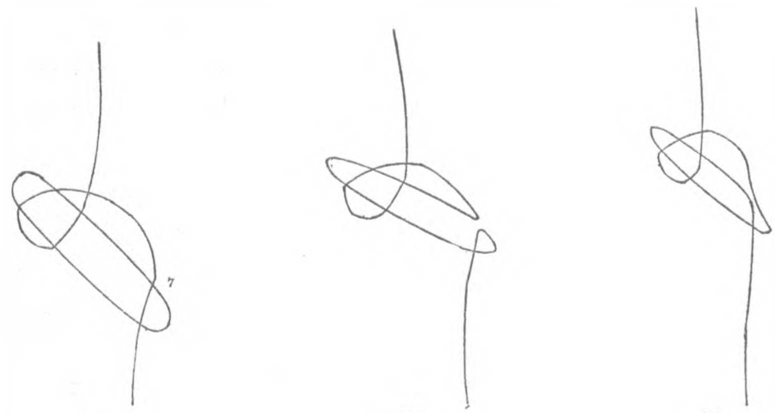
respectively. Then the equation:

$$\alpha_1(\phi_1^2 + \lambda_1^2\psi_1^2) + \varkappa_2\alpha_2(\phi_2^2 + \lambda_2^2\psi_2^2) + \varkappa_3\alpha_3(\phi_3^2 + \lambda_3^2\psi_3^2) = 0,$$

or

$$\alpha_1(\phi_1^2 + \psi_1^2) + \varkappa_2\alpha_2(\phi_2^2 + \psi_2^2) + \varkappa_3\alpha_3(\phi_3^2 + \psi_3^2) = 0,$$

represents any quintic curve having the given imaginary double points. In



(a) (b) (c)

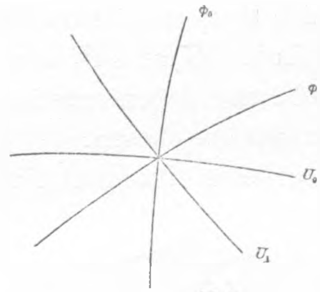
figure (a) the point 1 and any five of the intersections of ϕ_0 and U_0 may be taken as the six double points. Suppose the intersection numbered 7 is the one which is not a double point. If λ is taken very small the quintic will be very nearly of the form of U_0 and ϕ_0 except that it will not pass through the point 7. The form of the curve will be that given in figures (b) or (c) depending on the sign of λ. This shows that by taking a conic and cubic placed in the various possible relative positions and making a break as in the above figures, unicursal quintic curves result. This is the idea which has been used in constructing the curves given in this paper. By taking the conic through the node of the cubic and then proceeding as above, the forms of the curves having a triple point can be obtained. It is to be noticed that it is no restriction to always suppose that the cubic has but one infinite branch, provided the conic is not restricted.

(d)

Suppose figure (d) represents $\phi_1 U_1$ and that 8 is the point which is not a node. Then, as λ grows larger, the curve changes from that given by $U_0\phi_0$

(with a break at 7) to that given by $U_1\phi_1$ (with a break at 8). In the course of this change, each double point is successively a crunode, cusp, acnode, cusp, and finally, again a crunode when ϕ, U are not divided at the nodes. No attempt has been made to trace the curves through these changes. Four of the curves given by Meyer cannot be obtained by this method unless the conic is taken as a pair of lines. This does not mean that the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$ is lacking in generality, but that the four given curves correspond to intermediate values of λ. In the case of the triple point there is no difficulty of this kind, for it can be shown by the same methods that have been used here that if a straight line is drawn through one of the nodes of a trinodal quartic and a break made at one of the other intersections, a unicursal quintic curve with a triple point results. This gives the two curves which are not directly obtainable from the conic and cubic.

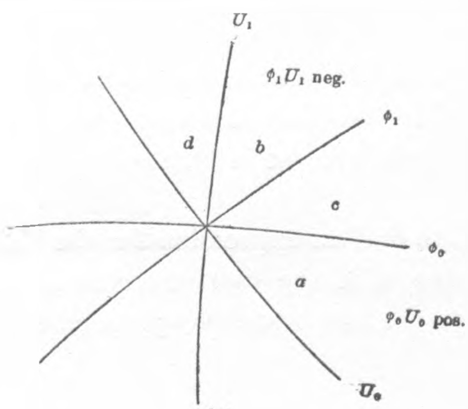
If $\phi_0 U_0$ and $\phi_1 U_1$ are divided at a given double point, as in figure (d_1), the



(d_1)

given point will always be a crunode no matter what value is given to λ; it may be positive or negative, large or small.

Suppose next the arrangement is that given in (e) and the products $\phi_0 U_0$



(e)

and $\phi_1 U_1$ are positive and negative in the compartments indicated. Then in the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$, if λ is negative, one branch of the curve is in (c) and the other in (d), while if λ is small and positive, the branches of the curve are both in (a), if large and positive, the branches are both in (b). In order that the curve shall change from one compartment to the other, the curve first has a cusp in compartment (a) (λ is now supposed a small positive quantity which is gradually growing larger), then there is a cusp in (b) and finally a crunode with the branches of the curve in compartment (b).

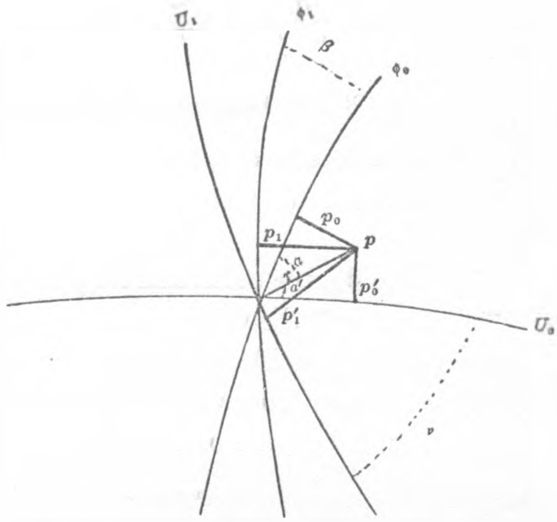
It might at first thought appear that it was possible for the curve to degenerate and have a tacnode instead of a cusp at the given point. Since the double points are fixed, there are only four possible ways of obtaining such a degenerate curve. To be definite, suppose the point considered is at 3, then the degenerate curve must be composed of a conic through the point 3, and any four of the remaining double points, together with a cubic which touches the conic at 3, has a node at the double point which does not lie on the conic, and passes through the four remaining double points. U_0 and U_1 each contain an arbitrary constant; clearly the quintic would not degenerate into the same conic and cubic independent of how these constants are chosen.

In case the part of the curve is like that given in figure (f), changing λ

(f)

so as to have a cusp at 4 implies that 1, 2 and 3 have been successively crunodes, cusps, acnodes and possibly the curve has come back in the other compartment, so that one or more of them have been changed back first to cusps and then to crunodes.

It is now proposed to find an expression for the limiting value of λ. For a very small positive value of λ the two branches of the curve will be close to $\phi_0 U_0$, but as λ increases, the two branches approach each other until the crunode is replaced by a cusp. Let p be such a point on the curve very close to the node, and let its coordinates be $x + \delta x,\ y + \delta y$.



Then $\phi_0 U_0 = \lambda \phi_1 U_1$,

$$\left(\delta x \frac{\partial \phi_0}{\partial x} + \delta y \frac{\partial \phi_0}{\partial y}\right)\left(\delta x \frac{\partial U_0}{\partial x} + \delta y \frac{\partial U_0}{\partial y}\right)$$
$$= \lambda \left(\delta x \frac{\partial \phi_1}{\partial x} + \delta y \frac{\partial \phi_1}{\partial y}\right)\left(\delta x \frac{\partial U_1}{\partial x} + \delta y \frac{\partial U_1}{\partial y}\right).$$

$$\lambda = \frac{\left(\delta x \frac{\partial \phi_0}{\partial x} + \delta y \frac{\partial \phi_0}{\partial y}\right)\left(\delta x \frac{\partial U_0}{\partial x} + \delta y \frac{\partial U_0}{\partial y}\right)}{\left(\delta x \frac{\partial \phi_1}{\partial x} + \delta y \frac{\partial \phi_1}{\partial y}\right)\left(\delta x \frac{\partial U_1}{\partial x} + \delta y \frac{\partial U_1}{\partial y}\right)}.$$

But
$$\delta x \frac{\partial \phi_0}{\partial x} + \delta y \frac{\partial \phi_0}{\partial y} = p_0 \sqrt{\left(\frac{\partial \phi_0}{\partial x}\right)^2 + \left(\frac{\partial \phi_0}{\partial y}\right)^2},$$
$$\delta x \frac{\partial U_0}{\partial x} + \delta y \frac{\partial U_0}{\partial y} = p_0' \sqrt{\left(\frac{\partial U_0}{\partial x}\right)^2 + \left(\frac{\partial U_0}{\partial y}\right)^2},$$
$$\delta x \frac{\partial \phi_1}{\partial y} + \delta y \frac{\partial \phi_1}{\partial y} = p_1 \sqrt{\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2},$$
$$\delta x \frac{\partial U_1}{\partial x} + \delta y \frac{\partial U_1}{\partial y} = p_1' \sqrt{\left(\frac{\partial U_1}{\partial x}\right)^2 + \left(\frac{\partial U_1}{\partial y}\right)^2};$$

$$p_0 = r \sin \alpha, \qquad p_0' = r \sin \alpha',$$
$$p_1 = r \sin(\alpha + \beta), \quad p_1' = r \sin(\alpha' + \gamma).$$

The required value is

$$\lambda = \frac{\sin\alpha \sin\alpha'}{\sin(\alpha+\beta)\sin(\alpha'+\gamma)} \frac{\left[\left(\frac{\partial\phi_0}{\partial x}\right)^2 + \left(\frac{\partial\phi_0}{\partial y}\right)^2\right]^{\frac{1}{2}} \cdot \left[\left(\frac{\partial U_0}{\partial x}\right)^2 + \left(\frac{\partial U_0}{\partial y}\right)^2\right]^{\frac{1}{2}}}{\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_1}{\partial y}\right)^2\right]^{\frac{1}{2}} \cdot \left[\left(\frac{\partial U_1}{\partial x}\right) + \left(\frac{\partial U_1}{\partial y}\right)^2\right]^{\frac{1}{2}}},$$

where $\dfrac{\sin\alpha \sin\alpha'}{\sin(\alpha+\beta)\sin(\alpha'+\gamma)}$ must be taken so as to be a maximum.

To obtain the form of the curves having two or more consecutive nodes it is only necessary to take the conic and cubic with the desired number of consecutive intersections.

Let the origin be taken at a double point and let $y=0$ be a common tangent to the two branches. The curve will then have at least two consecutive double points at the origin. The following table gives the expansion of the branches and the corresponding singularity. These expansions are given in Salmon's Higher Plane Curves, p. 216, including the oscnode and tacnode cusp.

Nature of the singularity.	Expansion.
$\delta = 4$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4, \\ y_2 = a_0x^2 + a_2x^3 + a_4'x^4, \end{cases}$
$\delta = 3,\ \varkappa = 1$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_5x^{\frac{9}{2}}, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_5'x^{\frac{9}{2}}, \end{cases}$
$\delta = 5$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6'x^5, \end{cases}$
$\delta = 4,\ \varkappa = 1$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_7x^{\frac{11}{2}}, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_7'x^{\frac{11}{2}}, \end{cases}$
$\delta = 6$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8x^6, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8'x^6, \end{cases}$
$\delta = 5,\ \varkappa = 1$	$\begin{cases} y_1 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8x^6 + a_9x^{\frac{13}{2}}, \\ y_2 = a_0x^2 + a_2x^3 + a_4x^4 + a_6x^5 + a_8x^6 + a_9'x^{\frac{13}{2}}. \end{cases}$

The curves having two or four imaginary nodes are obtained in exactly the same way as those having only real nodes, the only difference being that the conic is now taken so that in the first case two and in the second case four of its intersections with the cubic are imaginary.

As far as the sequence of double points is concerned, there is only one curve having six imaginary nodes. If in the equation

$$\alpha_1(\phi_1^2 + \psi_1^2) + \varkappa_2\alpha_2(\phi_2^2 + \psi_2^2) + \varkappa_3\alpha_3(\phi_3^2 + \psi_3^2) = 0,$$

x_2 and x_3 be taken very small, the form of the corresponding curve is very nearly the same as that of the straight line α_1.

2.—*Curves with a triple point.*

It will first be shown that the equation of any quintic curve with a triple point may be obtained from the special pencils

$$\alpha t^2 - \lambda\alpha_1 = 0, \quad \phi_1 t + \phi = 0, \tag{1}$$

or from

$$\alpha t^2 + \lambda\alpha_1 t - \alpha_1 = 0, \quad t\phi_1 + \phi = 0, \tag{2}$$

where t is a variable parameter, λ a constant, α and α_1 lines, ϕ and ϕ_1 conics.

On eliminating t from the first pair equations

$$\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0. \tag{3}$$

Similarly the equations (2) give

$$\alpha(\phi^2 - \phi_1^2) - \lambda\alpha_1\phi\phi_1 = 0. \tag{4}$$

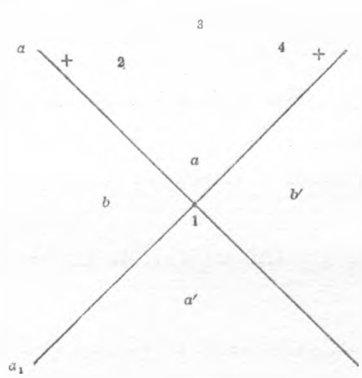
If in equations (3) α, $\lambda\alpha_1$, ϕ, ϕ_1, be replaced by $\alpha + i\lambda\alpha_1$, $\alpha - i\lambda\alpha_1$, $\phi + i\phi_1$, $\phi - i\phi_1$, respectively, the resulting equation reduces to

$$\alpha(\phi^2 - \phi_1^2) - 4\lambda\alpha_1\phi\phi_1 = 0, \tag{5}$$

which shows that (4) may be deduced from (3) or (3) from (4) by making the proper substitution.

It will now be shown that the equation $\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$ is the most general equation of a unicursal quintic curve having a triple point.

Let ϕ and ϕ_1 be two conics through the points 1, 2, 3, 4 and α, α_1 lines



through the point 1, λ a constant. Then the curve, whose equation is

$\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$, has a triple point at the point 1 and double points at 2, 3, 4. Moreover, each of the functions $\alpha, \alpha_1, \phi, \phi_1$ contains an undetermined constant which, together with λ, makes the number in the above equation 5, exactly that contained in the general equation of a unicursal quintic having a triple point.

Suppose in the above figure that there is an ordinary point of the curve in compartment a. Then the whole curve must lie in a (and a'), with the exception of possible acnodes. For, in order that a point shall be on the curve, $\alpha\phi^2$ must be equal to $\lambda\alpha_1\phi_1^2$ at the given point, but in crossing α or α_1 one of these terms changes sign and not the other, hence the only point at which it is possible for the curve to cross the lines α, α_1 is at their intersection. For the sake of definiteness, let λ be positive and let the positive sides of α and α_1 be taken as in the figure. Then if 2, 3 and 4 be taken in compartment a (or a'), they will be crunodes on the given quintic. By taking any or all of them in the compartment b (or b) or by simply changing the sign of λ, they become acnodes.

Take α through the point 2. Then, from the equation

$$\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0,$$

it is clear that since α and ϕ are both zero at 2, while α_1 is finite, points on the curve in the neighborhood of 2 must lie very close to ϕ_1. Moreover, since the curve has a double point at 2 and does not cross α, there must be a cusp at the given point with the tangent to ϕ_1 as the cuspidal tangent. By taking α_1 through one of the two remaining double points, it is possible to obtain another cusp.

If the conics intersect in only two real points, then the case arises in which the curve has two imaginary double points. These double points may become imaginary cusps, which would be the case in equation (4) if the conics ϕ and ϕ_1 are circles.

By taking conics having two consecutive intersections, a tacnode results, and if α is taken through the given point, the cusp is of the second kind. The conics might also be taken, having three or even all four of their intersections consecutive.

The curves represented by the equation $\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$ were drawn by fixing:

1st. The four points of intersection of ϕ and ϕ_1.

2d. The lines α and α_1.

3d. The three tangents at the triple point.

This makes a sufficient number of conditions to completely define the curve.

In drawing the curves, nothing more than the relative positions of these elements was considered, so that in a few cases the same initial conditions give rise to more than one curve. In such cases there would be some point at which the curve could move in different ways without violating the initial conditions.

The three double points were generally taken on the same side of the triple point. Of course, it is always possible to project the given curve into a curve in which this is the case, but it may not be possible to do so without altering the number of infinite branches. For instance, figs. 85, 112, 115, have the same relative positions of nodes, tangents at the triple point and tangents to the curve from the triple point. But if 85 is projected into a curve where the double points are situated as in 112, it will have at least three infinite branches: 115 may have its double points either way and still have only one infinite branch.

In some cases as for instance in figure 84 the curve may be tangent to α on either side of the triple point. This simply amounts to pushing the inflexion along the curve through the triple point, and although such curves are projectively distinct there is not enough difference to require two figures.

In drawing the curves represented by equation (4), the lines α and α_1 were not drawn but the tangents at the triple point were fixed as before and in addition the part of the curve going to infinity was taken in the various possible ways (joining two nodes, node to triple point, triple point to triple point). Use was also made of the fact that these curves can have no real tangents from the triple point, except those at the triple point.

The figures are only intended to give the general form of the curves. In some cases they do not show the total number of inflexions. It is then supposed that the curve has isolated double tangents.

The existence of these various types is also made plausible from Art. 1 by allowing three distinct double points to approach coincidence in various ways. The equations can also be determined exactly as in the preceding case, but it was frequently easier to follow the curve by present method.

3.—*Curves with a fourfold point.*

The case where the double points are all coincident would give but a single type according to Meyer's classification, but for these curves a different plan will be adopted, namely, the configuration of the fourfold point and the compartments formed by loops or the infinite branches will form the basis of classification. Here the reality of the inflexions will be considered.

The curve may be considered as generated by the intersection of corresponding rays of the pencils

$$a_0t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0$$

and

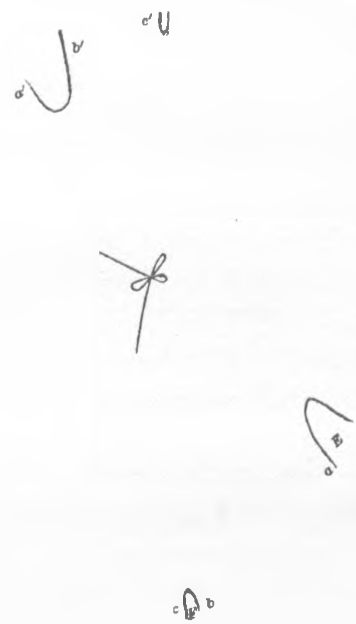
$$b_0t + b_1 = 0,$$

in which $a_i = 0$ and $b_i = 0$ are the equations of straight lines and t is a parameter. The vertex of the linear pencil is a fourfold point on the quintic: it will be taken as the origin so that the equation of the curve may be written $u_5 + u_4 = 0$. The factors of u_5 determine the directions of the infinite branches and the factors of u_4 determine the tangents at the origin. These tangents divide the plane into compartments such that the curve in passing from one compartment to another must either pass through the origin or through infinity. The nature of the fourfold point will depend on the form of u_4.

The following table gives Plücker's numbers and also $I + 2T$ for the various forms of the fourfold point.

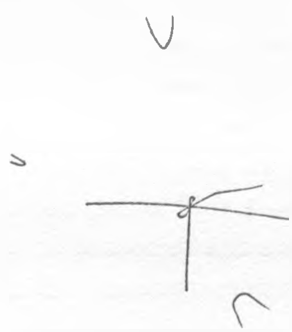
Factors of u_4	n	δ	$\varkappa$	τ	i	$I + 2T$	Figures.
$m_1 m_2 m_3 m_4$	8	6	0	12	9	3	134, 135, 136, 137,
$m_1^2 m_2 m_3$	7	5	1	8	7	3	138, 139, 140, 141, 142,
$m_1^2 m_2^2$	6	4	2	5	5	3	143, 144, 145,
$m_1^3 m_2$	6	4	2	5	5	3	146, 147, 148,
m_1^4	5	3	3	3	3	3	149,
$m_1 m_2 (m_3^2 + m_4^2)$	8	6	0	12	9	5	150, 151, 152, 153, 154, 155,
$m_1^2 (m_2^2 + m_3^2)$	7	5	1	8	7	5	156, 157, 158,
$(m_1^2 + m_2^2)(m_3^2 + m_4^2)$	8	6	0	12	9	5	159, 160,
$(m_1^2 + m_2^2)^2$	6	4	2	5	5	5	161, 162.

In case the factors of U_4 are real and distinct, there are the four types of curves represented in the first four figures. That the three inflexions may all be in one compartment formed by two tangents at the multiple point may be seen by considering a curve having five infinite branches in the given compartment as in the accompanying figure.



When the asymptotes are drawn, it is clear that there must be an inflexion somewhere along the curve either at a or a', one at b or b', and one at c or c'.

By taking a vanishing line close to the common tangent of E and F, this figure projects into figure 135. By taking four infinite branches in a given com-



partment, it is seen that the resulting curve has a loop which cuts the vanishing line in four points. The curve, therefore, has a loop with a bay. Figure 136 differs from figure (g) only in the position of the vanishing line.

The equations have not been written out for each case as they follow directly; for instance, the equation of figure 137 is

$$abc\,(d^2 + f^2) - \lambda\alpha\beta\gamma\delta = 0,$$

d and f are lines and λ is a parameter; a, b, c are lines through the origin parallel to the asymptotes, α, β, γ, δ are the four tangents to the curve at the fourfold point. The equation corresponding to a curve like figure 136 would not follow quite so directly, as it would be necessary to take the equation corresponding to figure (g), and perform on it the transformation corresponding to the projection of figure (g) into figure 136. No figures are drawn where the curve has five asymptotes, as such a curve can always be projected into one of the given figures.

In figures 143 to 148 the fourfold point counts as two cusps and four crunodes. In the first three figures the form of the curve indicates this, while in the last three the origin has the appearance of an ordinary double point. The presence of evanescent loops being shown by deformation.

Figure 155 differs from the others in that it contains a double bay. The possibility of this is readily seen from 141 (provided one of the loops in 141 had a bay) which was obtained from figure 136 by making two of the factors of u_4 equal. If now u_4 be changed still further so that these two factors become conjugate imaginaries, the cusp is replaced by an acnode and the curve takes the form of figure 155.*

When the curve has an isolated fourfold point, it is not possible to have fewer than three inflexions, as was shown by Möbius: "No odd branch containing no real point singularities can have fewer than three real inflexions."

It may be mentioned that there exists for every order n a curve corresponding to figure 137, i. e., having an $n-1$ fold point and $n-2$ infinite branches. For, take a curve having an $n-1$ fold point and whose equation can therefore be written $u_n + u_{n-1} = 0$. Take the factors of u_{n-1} real and distinct. The lines will divide the plane in $n-1$ compartments. If the lines represented by u_n are

* There should be given two figures whose form is the same as 141 excepting as regards inflexions, the one having a bay on one loop and the other having 3 real inflexions but none on the loop.

PLATE I.

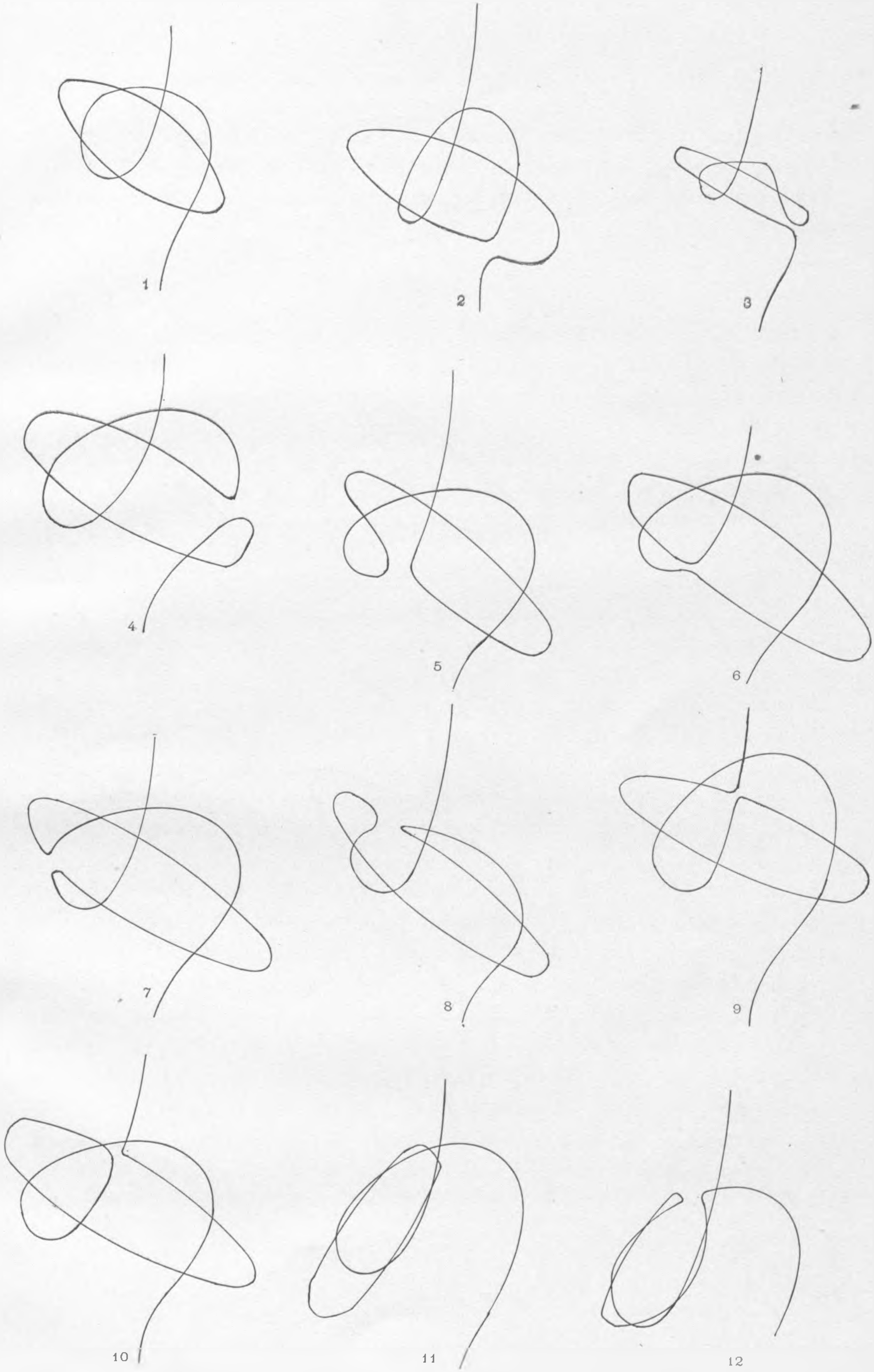


PLATE II.

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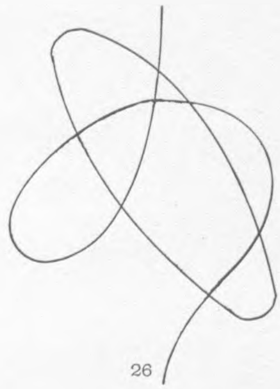
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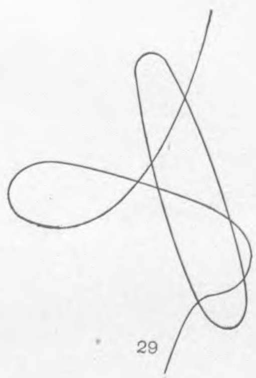
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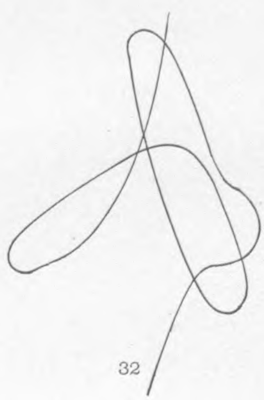
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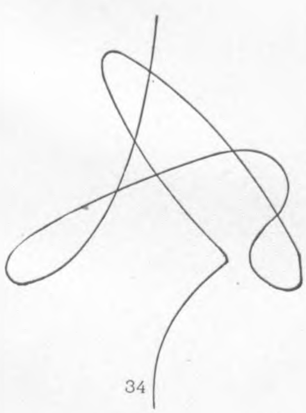
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PLATE IV.

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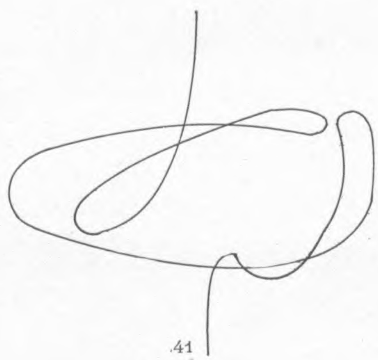

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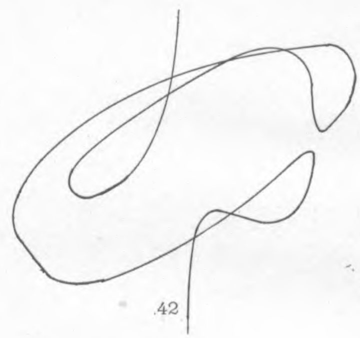

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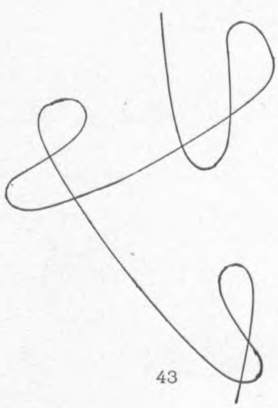

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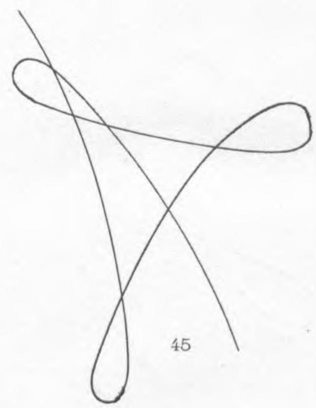

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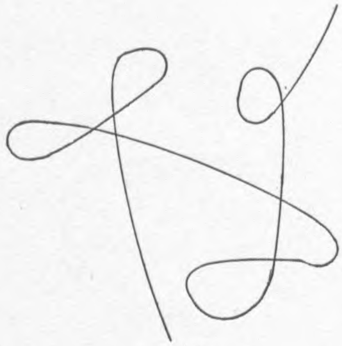

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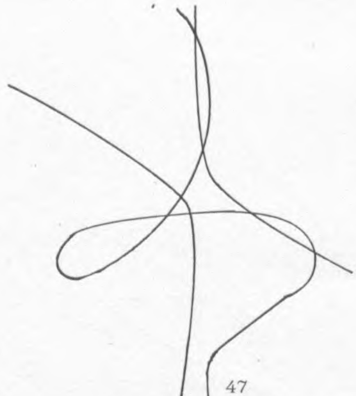

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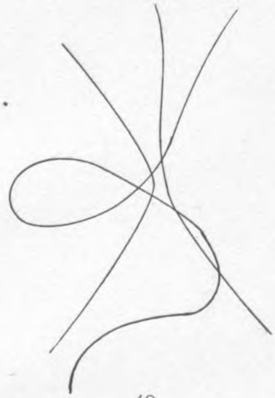
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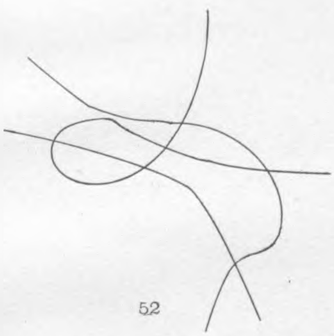
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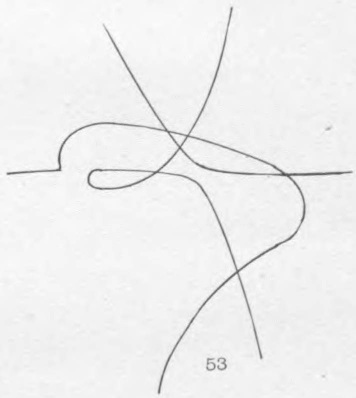
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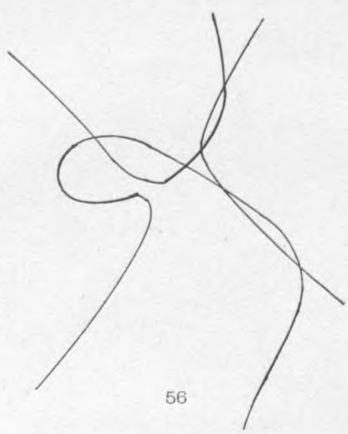
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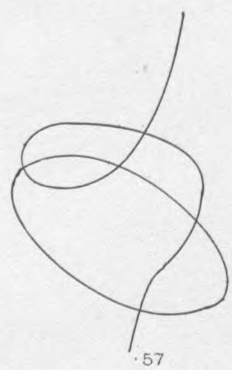
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PLATE VI.


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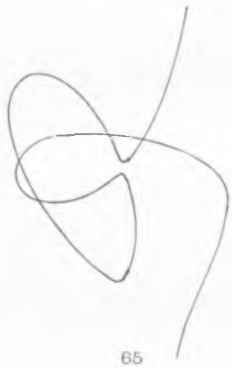
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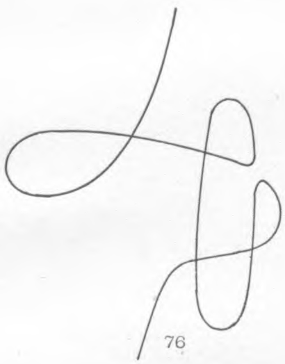

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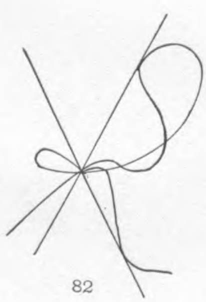

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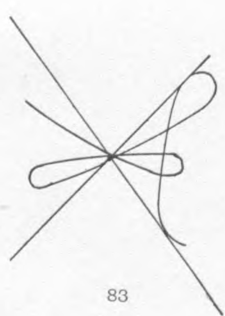

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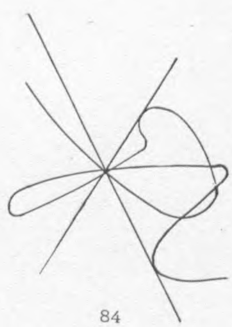

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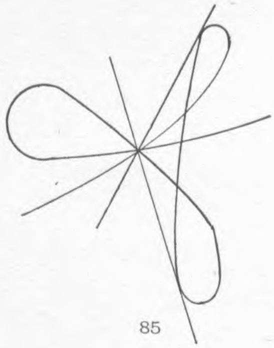
PLATE VIII.



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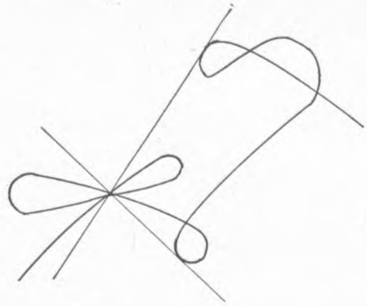

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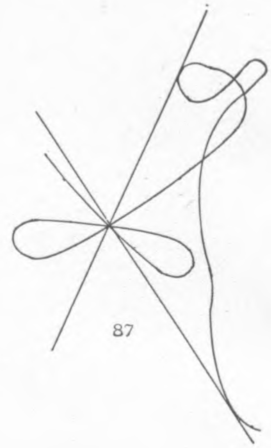

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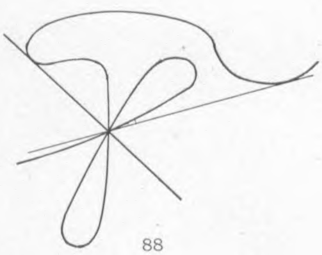

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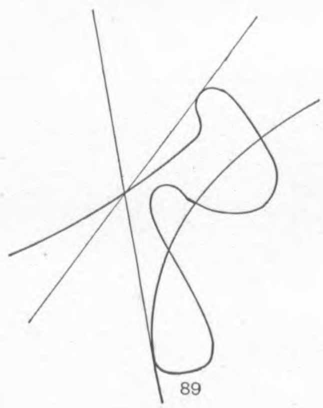

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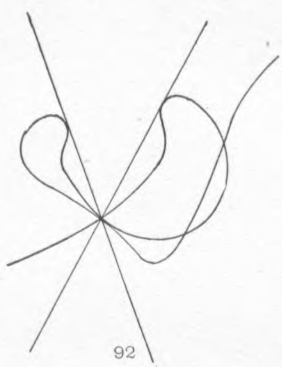

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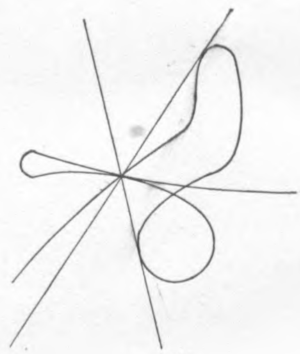

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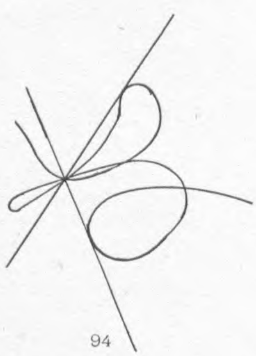

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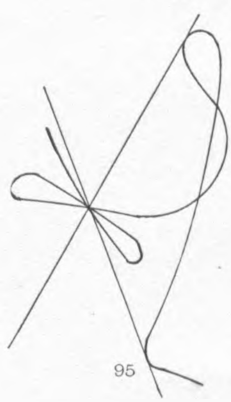

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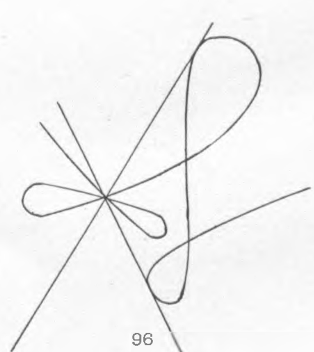

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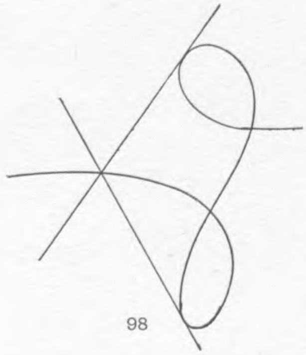
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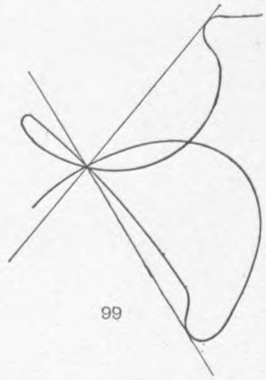
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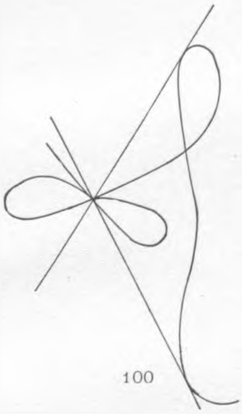
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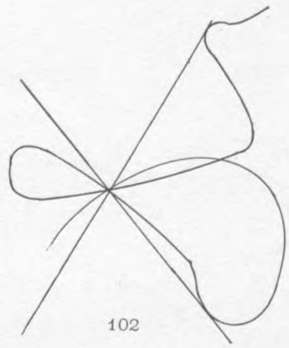
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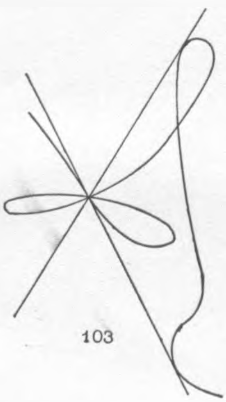
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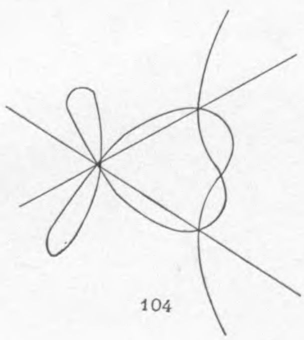
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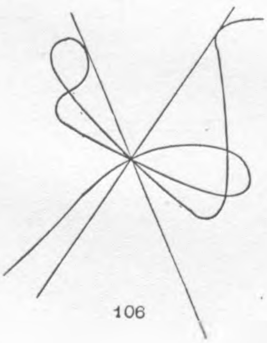
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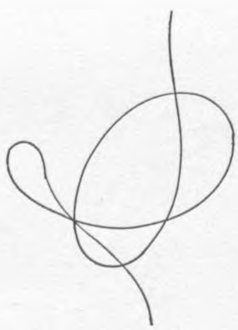
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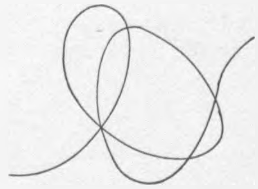
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PLATE X.

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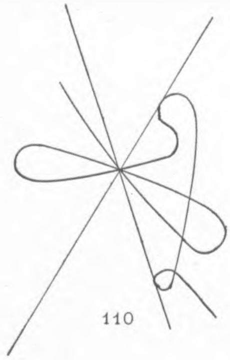

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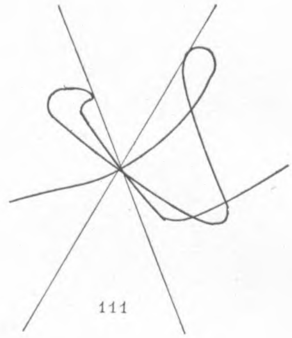

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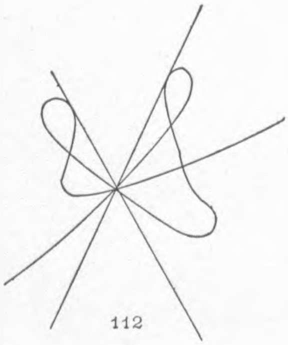

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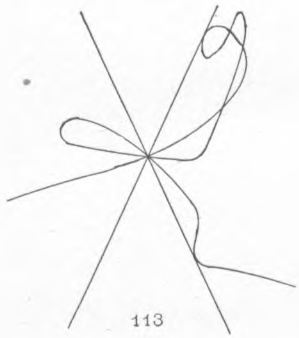

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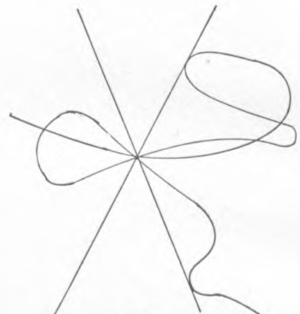

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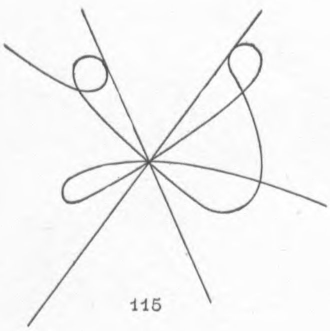

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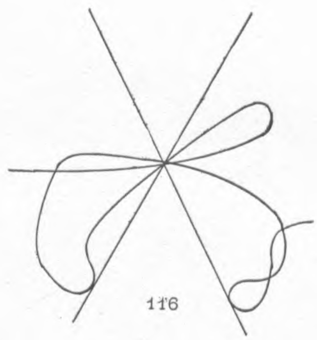

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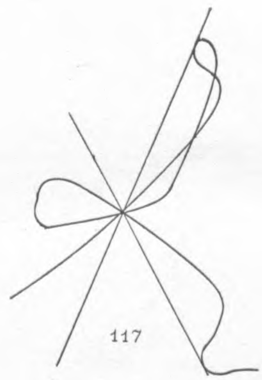

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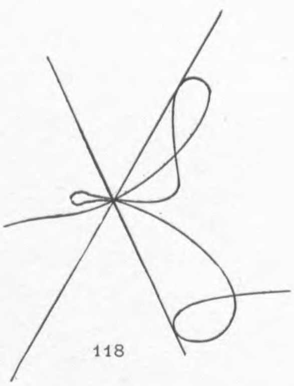

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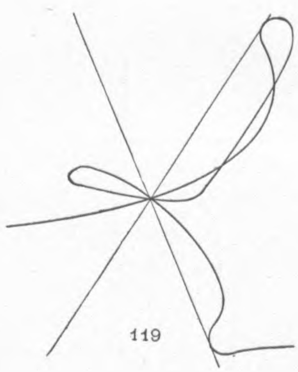

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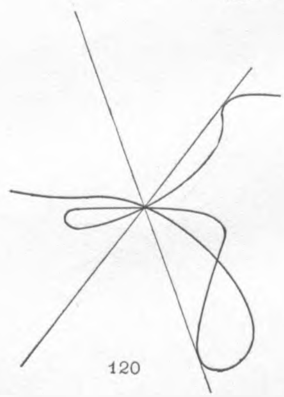

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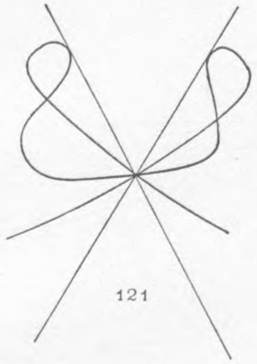

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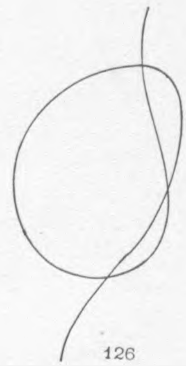

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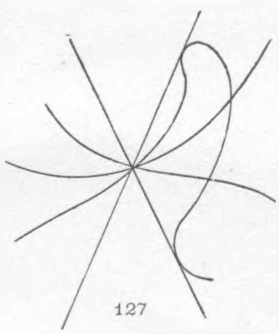

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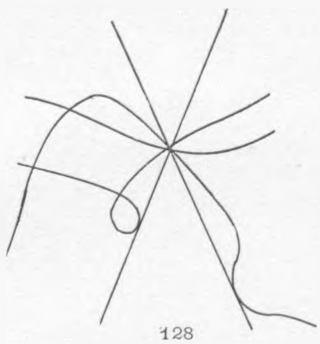

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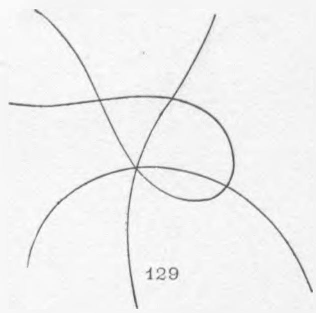

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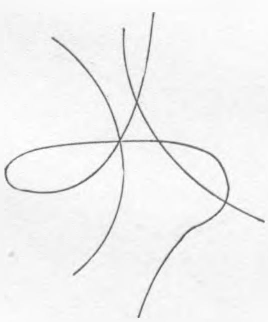

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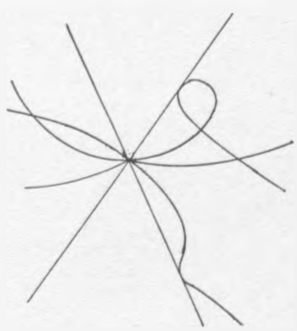

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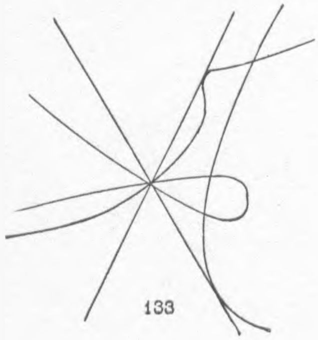
PLATE XII.



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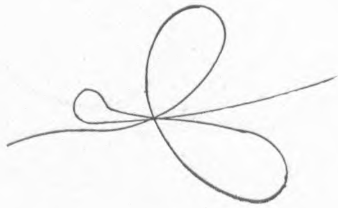

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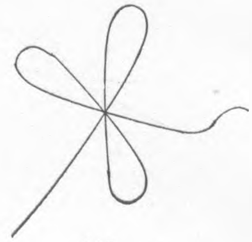

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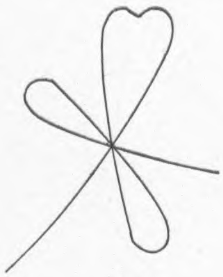

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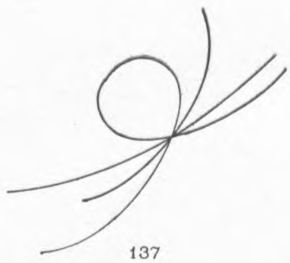

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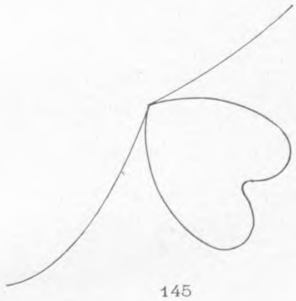

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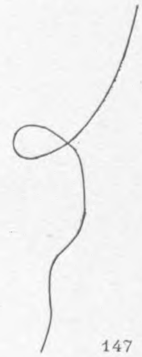

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PLATE XIV.

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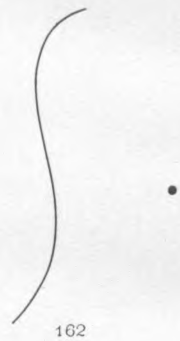

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taken so that one lies in each but one of the compartments, the resulting curve cannot be projected into one with fewer than $n - 2$ infinite branches, as the given curve has $n - 2$ branches that extend across the plane. The two remaining factors of u_n may be taken conjugate imaginary.

I wish to take this opportunity to express my obligation to the Department of Mathematics, particularly to Dr. Snyder, under whose supervision this work has been done.

CORNELL UNIVERSITY, *June*, 1902.